P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Using a Function Let $L(x) = \int_1^x \frac{1}{t}\,dt,\ x > 0.$

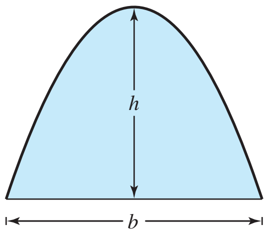
(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use a graphing utility to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Prove that $L(x_1 x_2) = L(x_1) + L(x_2)$ for all positive values of x_1 and x_2.

2. Parabolic Arch Archimedes showed that the area of a parabolic arch is equal to $\frac{2}{3}$ the product of the base and the height (see figure).



(a) Graph the parabolic arch bounded by $y = 9 - x^2$ and the x-axis. Use an appropriate integral to find the area A.

(b) Find the base and height of the arch and verify Archimedes' formula.

(c) Prove Archimedes' formula for a general parabola.

Evaluating a Sum and a Limit **In Exercises 3 and 4, (a) write the area under the graph of the given function defined on the given interval as a limit. Then (b) evaluate the sum in part (a), and (c) evaluate the limit using the result of part (b).**

3. $y = x^4 - 4x^3 + 4x^2, \quad [0, 2]$

$$\left(\textit{Hint: } \sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\right)$$

4. $y = \frac{1}{2}x^5 + 2x^3, \quad [0, 2]$

$$\left(\textit{Hint: } \sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}\right)$$

5. Fresnel Function The **Fresnel function** S is defined by the integral

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

(a) Graph the function $y = \sin\left(\frac{\pi x^2}{2}\right)$ on the interval $[0, 3]$.

(b) Use the graph in part (a) to sketch the graph of S on the interval $[0, 3]$.

(c) Locate all relative extrema of S on the interval $(0, 3)$.

(d) Locate all points of inflection of S on the interval $(0, 3)$.

6. Approximation The **Two-Point Gaussian Quadrature Approximation** for f is

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

(a) Use this formula to approximate

$$\int_{-1}^{1} \cos x\,dx.$$

Find the error of the approximation.

(b) Use this formula to approximate

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx.$$

(c) Prove that the Two-Point Gaussian Quadrature Approximation is exact for all polynomials of degree 3 or less.

7. Extrema and Points of Inflection The graph of the function f consists of the three line segments joining the points $(0, 0)$, $(2, -2)$, $(6, 2)$, and $(8, 3)$. The function F is defined by the integral

$$F(x) = \int_0^x f(t)\,dt.$$

(a) Sketch the graph of f.

(b) Complete the table.

x	0	1	2	3	4	5	6	7	8
$F(x)$									

(c) Find the extrema of F on the interval $[0, 8]$.

(d) Determine all points of inflection of F on the interval $(0, 8)$.

8. Falling Objects Galileo Galilei (1564–1642) stated the following proposition concerning falling objects:

> *The time in which any space is traversed by a uniformly accelerating body is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed of the accelerating body and the speed just before acceleration began.*

Use the techniques of this chapter to verify this proposition.

9. Proof Prove $\int_0^x f(t)(x-t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right) dt.$

10. Proof Prove $\int_a^b f(x)f'(x)\,dx = \frac{1}{2}([f(b)]^2 - [f(a)]^2).$

11. Riemann Sum Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n^{3/2}}.$$

12. Riemann Sum Use an appropriate Riemann sum to evaluate the limit

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}.$$

13. Proof Suppose that f is integrable on $[a, b]$ and $0 < m \le f(x) \le M$ for all x in the interval $[a, b]$. Prove that

$$m(a - b) \le \int_a^b f(x)\,dx \le M(b - a).$$

Use this result to estimate $\int_0^1 \sqrt{1 + x^4}\,dx$.

14. Using a Continuous Function Let f be continuous on the interval $[0, b]$, where $f(x) + f(b - x) \ne 0$ on $[0, b]$.

(a) Show that $\displaystyle\int_0^b \frac{f(x)}{f(x) + f(b - x)}\,dx = \frac{b}{2}$.

(b) Use the result in part (a) to evaluate

$$\int_0^1 \frac{\sin x}{\sin(1 - x) + \sin x}\,dx.$$

(c) Use the result in part (a) to evaluate

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3 - x}}\,dx.$$

15. Velocity and Acceleration A car travels in a straight line for 1 hour. Its velocity v in miles per hour at six-minute intervals is shown in the table.

t (hours)	0	0.1	0.2	0.3	0.4	0.5
v (mi/h)	0	10	20	40	60	50

t (hours)	0.6	0.7	0.8	0.9	1.0
v (mi/h)	40	35	40	50	65

(a) Produce a reasonable graph of the velocity function v by graphing these points and connecting them with a smooth curve.

(b) Find the open intervals over which the acceleration a is positive.

(c) Find the average acceleration of the car (in miles per hour squared) over the interval $[0, 0.4]$.

(d) What does the integral

$$\int_0^1 v(t)\,dt$$

signify? Approximate this integral using the Trapezoidal Rule with five subintervals.

(e) Approximate the acceleration at $t = 0.8$.

16. Proof Prove that if f is a continuous function on a closed interval $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

17. Verifying a Sum Verify that

$$\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

by showing the following.

(a) $(1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $(n + 1)^3 = \displaystyle\sum_{i=1}^{n} (3i^2 + 3i + 1) + 1$

(c) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$

18. Sine Integral Function The **sine integral function**

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is often used in engineering. The function

$$f(t) = \frac{\sin t}{t}$$

is not defined at $t = 0$, but its limit is 1 as $t \to 0$. So, define $f(0) = 1$. Then f is continuous everywhere.

(a) Use a graphing utility to graph $\mathrm{Si}(x)$.

(b) At what values of x does $\mathrm{Si}(x)$ have relative maxima?

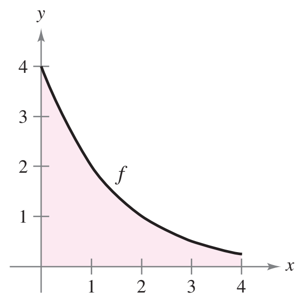
(c) Find the coordinates of the first inflection point where $x > 0$.

(d) Decide whether $\mathrm{Si}(x)$ has any horizontal asymptotes. If so, identify each.

19. Comparing Methods Let

$$I = \int_0^4 f(x)\,dx$$

where f is shown in the figure. Let $L(n)$ and $R(n)$ represent the Riemann sums using the left-hand endpoints and right-hand endpoints of n subintervals of equal width. (Assume n is even.) Let $T(n)$ and $S(n)$ be the corresponding values of the Trapezoidal Rule and Simpson's Rule.



(a) For any n, list $L(n)$, $R(n)$, $T(n)$, and I in increasing order.

(b) Approximate $S(4)$.

20. Minimizing an Integral Determine the limits of integration where $a \le b$ such that

$$\int_a^b (x^2 - 16)\,dx$$

has minimal value.